The Entanglement in Anisotropic Heisenberg XYZ Chain with inhomogeneous magnetic field

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The thermal entanglement of a two-qubit anisotropic Heisenberg XYZ chain under an inhomogeneous magnetic field b is studied. It is shown that when inhomogeneity is increased to certain value, the entanglement can exhibit a larger revival than that of less values of b. The property is both true for zero temperature and a finite temperature. The results also show that the entanglement and critical temperature can be increased by increasing inhomogeneous exteral magnetic field.

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I. INTRODUCTION

As a valuable resource in quantum information and quantum computation [1], quantum entanglement has attracted numerous attention over past decade years. In order to realize the quantum information process, a great deal of effort has been devoted to study and characterize the entanglement in solid state systems [2,3]. A typical example in solid system is the spin chains which are the natural candidates for the realization of the entanglement compared with the other physics systems. The Heiserberg model can describe interaction of qubits not only in solid physical systems but also in many other systems such as quantum dots [4] and nuclear spin [5]; therefore, there are numerous studies on the Heisenberg models[6-13]. It is turned out that the critical magnetic field B_c is decreased with the increase of the anisotropic parameter γ but the critical temperature T_c is improved. When B crosses B_c the concurrence C drops suddenly and then undergoes a "revival" for sufficiently large γ in Ref [14], but it only discussed the uniform magnetic field case. As we all know, in any solid state construction of qubits, there is always the possibility of inhomogeneous Zeeman coupling [15,16]. Quite recently the effect of inhomogeneous magnetic field on the thermal entanglement of an isotropic two-qubit XXX spin system has been studied in Ref [17]. Including interaction of Z-component J_z, Ref [18] also studied the effect of inhomogeneous magnetic field on the thermal entanglement in a two-qubit Heisenberg XXZ spin chain.

However, the entanglement for a XYZ spin model under an inhomogeneous magnetic field has not been discussed. Therefore, in this paper we investigate the influence of an external inhomogeneous magnetic field on the entanglement of a two-qubit Heisenberg XYZ system at thermal equilibrium. Our studies show that the inhomogenerous extral magnetic field can make the revival entanglement larger,improve the critical temperature, and enhance entanglement.

II. THEORETICAL TREATMENT AND RESULTS

The Heisenberg Hamiltonian of a N-qubit anisotropic Heisenberg XYZ model under an inhomogeneous magnetic field is

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^{N}[J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z \\ &\quad +(B+b)\sigma_i^z + (B-b)\sigma_{i+1}^z], \end{aligned} \tag{1}$$

where (σ_i^x, σ_i^y, σ_i^z) are the vector of Pauli matrices and J_j $(j = x, y, z)$ is the anisotropic coupling coefficient between the nearest two spins. The parameter $J_j > 0$ means that the chain is antiferromagnetic, and ferromagnetic for $J_j < 0$. The magnetic fields on the nearest-neighbor two qubits are $B - b$ and $B + b$, respectively, the value of b controls the degree of inhomogeneity.

For a spin system in equilibrium at temperature T, the density matrix is $\rho = (1/Z)\exp(-H/k_BT)$, where H is the Hamiltonian of this system, Z is the partition function and k_B is the Boltzmann constant. Usually we write $k_B = 1$. For a two-qubit system the thermal entanglement can be measured by the concurrence C which can be calculated with the help of $Wootters'$ formula[19] $C = \max(0, 2\max\lambda_i - \sum_{i=1}^{4}\lambda_i)$, where λ_i is the square roots of the eigenvalues of the matrix

$$R = \rho(\sigma_1^y \otimes \sigma_2^y)\rho^*(\sigma_1^y \otimes \sigma_2^y), \tag{2}$$

where the asterisk indicates complex conjugation, the concurrence C ranges from zero to one.

Consider now the Hamiltonian H for the anisotropic two-qubit Heisenberg XYZ chain in an inhomogeneous magnetic field. The Hamiltonian can be shown as

$$\begin{aligned} H &= J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + J\gamma(\sigma_1^+\sigma_2^+ + +\sigma_1^-\sigma_2^-) + \frac{J_z}{2}\sigma_1^z\sigma_2^z \\ &\quad +\frac{(B+b)}{2}\sigma_1^z + \frac{(B-b)}{2}\sigma_2^z, \end{aligned} \tag{3}$$

where $J = \frac{(J_x+J_y)}{2}$, $\gamma = \frac{J_x-J_y}{J_x+J_y}$ and $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. Among these parameters $\sigma^{\pm}$ are raising and lowering

operators respectively and γ ($0 < \gamma < 1$) measures the anisotropy in the XY plane. In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the Hamiltonian can be expressed as

$$H = \begin{pmatrix} \frac{J_z}{2}+B & 0 & 0 & J\gamma \\ 0 & -\frac{J_z}{2}+b & J & 0 \\ 0 & J & -\frac{J_z}{2}-b & 0 \\ J\gamma & 0 & 0 & \frac{J_z}{2}-B \end{pmatrix}.$$

The eigenvectors and eigenvalues of H are easily obtained as following forms $H|\psi^{\pm}\rangle = (-\frac{J_z}{2} \pm \xi)|\psi^{\pm}\rangle$ and $H|\Sigma^{\pm}\rangle = (\frac{J_z}{2} \pm \eta)|\Sigma^{\pm}\rangle$, with the eigenstates are $|\psi^{\pm}\rangle = N^{\pm}[\frac{(b\pm\xi)}{J}|01\rangle + |10\rangle]$ and $|\Sigma^{\pm}\rangle = M^{\pm}[\frac{(B\pm\eta)}{J\gamma}|00\rangle + |11\rangle]$, where $\eta = \sqrt{B^2+J^2\gamma^2}$ and $\xi = \sqrt{b^2+J^2}$, the normalization constants are $N^{\pm} = 1/\sqrt{1+(b\pm\xi)^2/J^2}$ and $M^{\pm} = 1/\sqrt{1+(B\pm\eta)^2/J^2\gamma^2}$. One can notice that these four eigenstates are all entanglement states when $J \neq 0$ so it means that entanglement exists for both antiferromagnetic ($J > 0$) and ferromagnetic ($J < 0$) cases.

Now we can calculate the thermal entanglement of this system. The density matrix can be written as the following form

$$\rho = \begin{pmatrix} \mu_+ & 0 & 0 & \upsilon \\ 0 & \omega_1 & z & 0 \\ 0 & z & \omega_2 & 0 \\ \upsilon & 0 & 0 & \mu_- \end{pmatrix}.$$

The square roots of the eigenvalues of the matrix R are

$$\lambda_{1,2} = |\sqrt{\mu_+\mu_-} \pm \upsilon|, \lambda_{3,4} = |\sqrt{\omega_1\omega_2} \pm z|. \tag{4}$$

The exact values of these nonzero matrix elements can be obtained by knowing the spectrum of H. We obtain

$$\mu_+ = \frac{1}{Z}e^{-\frac{J_z}{2}\beta}[\cosh(\eta\beta) - \frac{\beta}{\eta}\sinh(\eta\beta)],$$

$$\mu_- = \frac{1}{Z}e^{-\frac{J_z}{2}\beta}[\cosh(\eta\beta) + \frac{\beta}{\eta}\sinh(\eta\beta)],$$

$$z = -\frac{J}{Z\xi}e^{\frac{J_z}{2}\beta}\sinh(\xi\beta),$$

$$\omega_1 = \frac{1}{Z}e^{\frac{J_z}{2}\beta}[\cosh(\xi\beta) - \frac{b}{\xi}\sinh(\xi\beta)],$$

$$\omega_2 = \frac{1}{Z}e^{\frac{J_z}{2}\beta}[\cosh(\xi\beta) + \frac{b}{\xi}\sinh(\xi\beta)],$$

$$\upsilon = -\frac{J\gamma}{Z\eta}e^{-\frac{J_z}{2}\beta}\sinh(\eta\beta), \tag{5}$$

where Z is the partition function given by

$$Z = tre^{-\beta H} = 2[e^{-\frac{J_z}{2}\beta}\cosh(\eta\beta) + e^{\frac{J_z}{2}\beta}\cosh(\xi\beta)]. \tag{6}$$

Thus from Eq.(4)-(6) we find that

$$\lambda_{1,2} = \frac{1}{Z}e^{-\frac{J_z}{2}\beta}|\sqrt{1+\frac{J^2\gamma^2}{\eta^2}\sinh(\eta\beta)^2} \mp \frac{J\gamma}{\eta}\sinh(\eta\beta)|,$$

$$\lambda_{3,4} = \frac{1}{Z}e^{\frac{J_z}{2}\beta}|\sqrt{1+\frac{J^2}{\xi^2}\sinh(\xi\beta)^2} \mp \frac{J}{\xi}\sinh(\xi\beta)|. \tag{7}$$

We can calculate the thermal entanglement which is measured by the definition of the concurrence. One can notice that Eq.(7) are the same as the case of $b = 0$ [20]. The concurrence which is directly determined by $\lambda_i (i = 1,2,3,4)$ is invariant under the substitutions $\gamma \longrightarrow -\gamma$ and $J \longrightarrow -J$ but the eigenvalue is variant with the substitution $J_z \longrightarrow -J_z$ so that the concurrence is also different. We will take the cases $J > 0$ and $0 \leq \gamma \leq 1$ into account in this paper.

From Eq.(7) and the definition of the concurrence, we derive the C for $T = 0$ as following

$$C(T=0) = \begin{cases} \frac{J\gamma}{\eta} & \xi < \eta - J_z, \\ |\frac{J\gamma}{\eta} - \frac{J}{\xi}|/2 & \xi = \eta - J_z, \\ \frac{J}{\xi} & \xi > \eta - J_z. \end{cases} \tag{8}$$

Although the parameters J, γ, η and ξ are independent of J_z in the case of two interacting qubits, the value of J_z is very important in determining the point of the piecewise function so that it can play role in the pairwise entanglement. We will show it in Fig.1.

The concurrence C as a function of b at $T = 0$ for three values of γ are given in Fig.1. Consider first the nonuniform magnetic field ($B = 0$) case [Fig. 1(a)]. With increasing b, the concurrence C is initially constant and equal to its maximal value 1. It then drops suddenly as a critical value b_c is reached, where the critical inhomogeneous magnetic field b_c is given by $\xi = \eta - J_z$ or $b_c = \sqrt{(\eta - J_z)^2 - J^2}$. At the critical point ($T = 0$, $b = b_c$), the entanglement becomes a nonanalytic function of b and a quantum phase transition occurs, but for $b > b_c$ the concurrence C undergoes a revival before decreasing to zero. The critical inhomogeneous magnetic field b_c is increased by increasing the anisotropy parameter γ, which means the region that entanglement keeps its maximum value is broaden with the increasing γ. However, for $B \neq 0$ case, entanglement exhibits novel property shown in Fig. 1(b). For certain relationship of the parameters, the value of revival can be larger than its value before dropping, for example, Fig1.(b) $\gamma = 0.2$ curve. In addition, the region of entanglement keeping constant and entanglement values is increased with the increasing γ. In Fig.1 (c), we further show the role of J_z in existing larger revival. One can see clearly the values

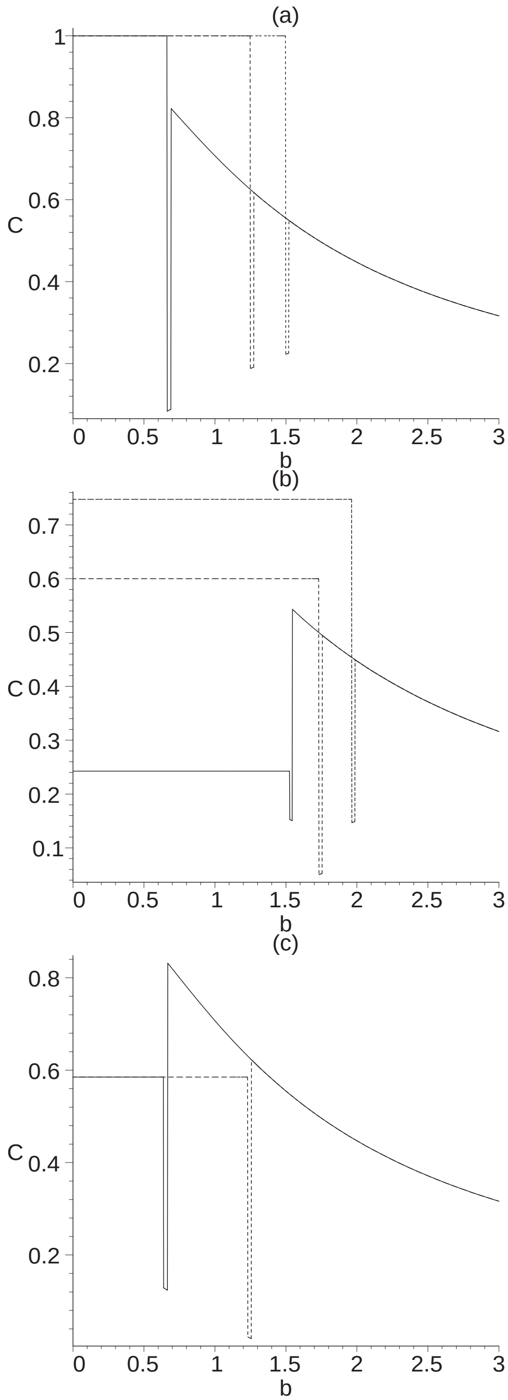

FIG. 1: The concurrence in the XYZ model is plotted vs b for various values of anisotropy parameter. (a):From left to right $\gamma = 0.2$, 0.6 ,0.9, respectively, where $B = 0$ and $J_z = -1$. (b): From bottom to up $\gamma = 0.2, 0.6, 0.9$, respectively, $B = 0.8$ and $J_z = -1$. (c) gives the concurrence with different values of J_z: $J_z = -0.2$ (solid line), $J_z = -0.6$ (dashed line) with $\gamma = 0.6$ and $B = 0.8$. For all plotted $T = 0$ and $J = 1$.

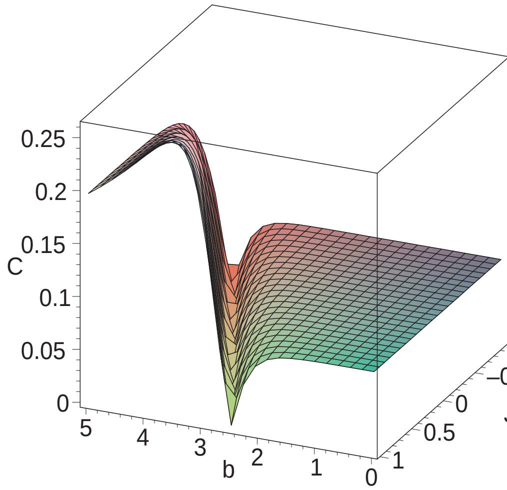

FIG. 2: (color online) Concurrence versus the inhomogeneous magnetic field field b and J_z, where $J = 1$, $B = 4$, $T = 0.2$ and $\gamma = 0.3$.

of J_z broaden the region of entanglement keeping constant values. The most important is that only when J_z achieve certain values, the value of entanglement revival can larger than that of before dropping.

Now we discuss the conditions of existing revival and larger revival. From Eq.(8), we see if $\xi \geqslant \eta - J_z$, i.e., $b_c \geqslant \sqrt{(\eta - J_z)^2 - J^2}$, the concurrence C will exhibit revival phenomenon. But if we need larger revival value, combining $\frac{J}{\xi} > \frac{J\gamma}{\eta}$ and $\xi \geqslant \eta - J_z$, we have $J_z > \eta - \frac{\eta}{\gamma}$. Therefore, the condition existing larger revival is as

$$\begin{cases} b_c \geqslant \sqrt{(\eta - J_z)^2 - J^2} \\ J_z > \eta - \frac{\eta}{\gamma}. \end{cases} \tag{9}$$

In figure 2 we give the plot of concurrence as a function of b and J_z at a fixed temperature and magnetic field for $\gamma = 0.3$. It is shown that the critical inhomogeneous magnetic field b_c is increased with the decrease of the interaction of z-component J_z and there is no entanglement at this critical point. As b increases, the concurrence C is initially constant which is nearly invariant by changing the interaction of z-component J_z. It then decreases suddenly as the critical inhomogeneous magnetic field b_c is reached. However, instead of vanishing for $b > b_c$, the concurrence persists and undergoes a revival before decreasing. The most important thing is that the value of entanglement revival can larger than that of before dropping. The maximal entanglement value exists in the revival region, which means we still can observe larger revival phenomenon for thermal entanglement.

With $B = 4$, the concurrence as a function of b and T are given in Fig.3. If b is lower than a certain value, there are two areas showing entanglement, and the entanglement first decreases and then undergoes a revival before decreasing to zero. But when b is larger than this certain value, there is no revival phenomenon and the entanglement is decreased monotonously with the increase of T.

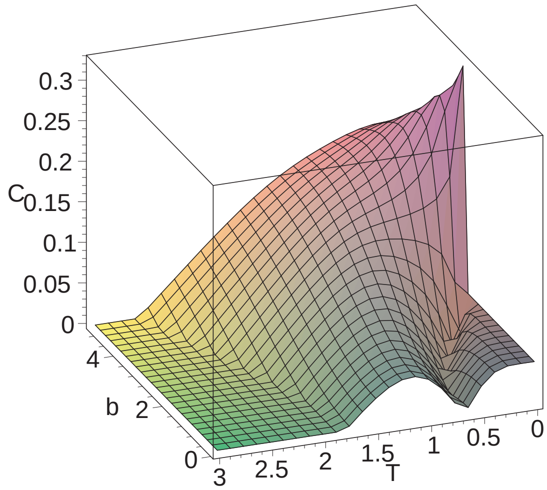

FIG. 3: (color online) Concurrence versus temperature and inhomogeneity for $\gamma = 0.2$ and $B = 4$. The coupling constant $J = 1$ and $J_z = 1$.

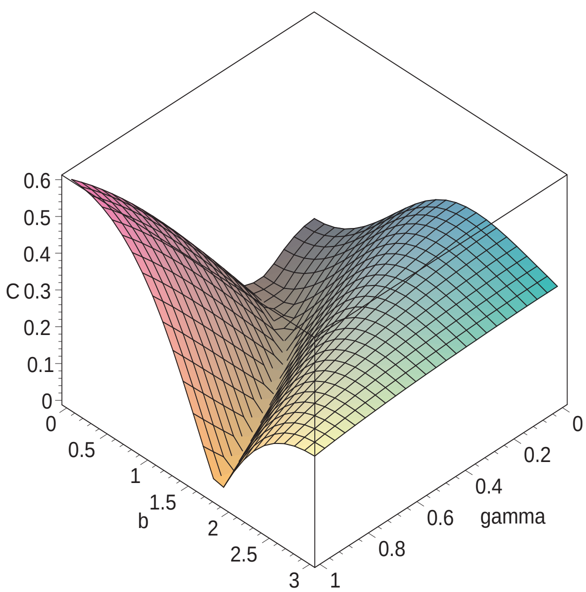

FIG. 4: (color online) The concurrence in the XYZ spin model is plotted vs b and anisotropy parameter γ, where $T = 0.4$ and $B = 0.8$. The coupling constant $J_z = -0.6$ and $J = 1$.

However, in the region of larger (b, T) parameter space, we can derive larger entanglement. In addition, a critical temperature T_c is improved with the increase of b. The property opens out the role of inhomogeneous magnetic field b in improving entanglement. Ref.[10] pointed out that a pairwise entanglement in N-qubit isotropic Heisenberg system in certain degree can be increased by introducing external field B. Our study shows the inhomogeneous magnetic field b can have a better action on improving entanglement for thermal entanglement.

Figure 4 shows the entanglement as measured by the concurrence in terms of the inhomogeneity b and anisotropy parameter γ at a finite temperature T for a finite external magnetic field B ($B = 0.8$). We can observe that entanglement exists in two regions: the one is in large γ and small b; the other is in small γ and b. If the anisotropy of XY plane γ is large enough, we just need small inhomogeneous external field b or even without inhomogeneity, vice versa. At this point, the role of b is something similar to the anisotropy XY plane. It seems to be possible to control the anisotropy by adjusting external magnetic field.

III. CONCLUSION

In conclusion, we have studied the thermal entanglement in an anisotropic two qubits Heisenberg XYZ chain under an inhomogeneous magnetic field. Through analyzing the $T = 0$ case, we find that conditions of existing revival phenomenon and larger revival. The larger revival phenomenon still exists at fixed finite temperature. If the parameters are proper and the inhomogeneity is large enough, in the revival region the entanglement is larger than that of without inhomogeneous magnetic field for fixed temperature. Then we show by increasing the inhomogeneous magnetic field b the entanglement and the critical temperature T_c can be improved.

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